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Engineering Notes

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Dynamics of a Particle Moving Along an Orbital Tower

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Introduction

THE concept of an orbital tower has been discussed in the literature by many authors over a number of years. Although the concept is clearly futuristic, interest has recently been revived as a result of advances in materials science (for example, see Refs. 1–4). In this Note, a simple model of the dynamics of a particle moving along an orbital tower is considered. First, it is demonstrated that at synchronous radius there exists a hyperbolic fixed point,[5] resulting in an unstable equilibrium and a potential barrier that a particle must cross. The fixed point is an equilibrium point in the phase space, which represents the dynamics of the particle. It is shown that the addition of friction does not remove the hyperbolic fixed point, but merely modifies its instability timescale. Finally, it is shown that friction leads to phase paths converging asymptotically to a single manifold in the phase space of the problem. An approximation to this manifold is constructed. The analysis provides some insight into the practical application of orbital towers for the launch and retrieval of payloads.

Free Particle Dynamics

A point mass is considered moving along a tether of length L, assuming that the tether is rigid and corotates with the Earth (radius R_E) at constant angular velocity Ω, as shown in Fig. 1. As the particle moves along the tether, it experiences a transverse Coriolis force resulting in friction between the particle and tether, which will be considered later. For the frictionless case, the effective potential ϕ is the sum of the gravitational potential of the Earth and a potential that represents the centripetal acceleration defined as

$$\phi(R) = -\tfrac{1}{2}R^2\Omega^2 - \mu/R \tag{1}$$

where μ is the gravitational parameter of the Earth and R is the distance of the particle from the center of the Earth. This one-dimensional problem is conservative and can be described by a two-dimensional phase space with Hamiltonian $\mathcal{H}(R, \dot{R})$ defined on $R \in \Re\backslash[0, R_E)$, $\dot{R} \in \Re$ and given by

$$\mathcal{H}(R, \dot{R}) = \tfrac{1}{2}\dot{R}^2 + \phi(R) \tag{2}$$

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Because the system is conservative, $\dot{\mathcal{H}}(R, \dot{R}) = 0$, and the problem immediately presents an integral of motion given by $\mathcal{H}(R, \dot{R}) = C$ so that

$$C = \tfrac{1}{2}\dot{R}^2 - \tfrac{1}{2}R^2\Omega^2 - \mu/R \tag{3}$$

This integral allows the phase space of the problem to be explored for level curves of C. Then, the equation of motion of the particle can be determined from the Hamiltonian using $\dot{R} = \partial\mathcal{H}/\partial\dot{R}$ and $\ddot{R} = -\partial\mathcal{H}/\partial R$, which yields

$$\ddot{R} = R\Omega^2 - \mu/R^2 \tag{4}$$

It is clear from Eq. (4) that a single equilibrium point exists when the radial acceleration vanishes at the point $R = \bar{R}$, defined by

$$\frac{\partial\mathcal{H}(R, \dot{R})}{\partial R} = 0 \Rightarrow \bar{R} = \left[\frac{\mu}{\Omega^2}\right]^{\frac{1}{3}} \tag{5}$$

which corresponds to a single equilibrium point E at synchronous radius (6.6 Earth radii). The nature of this equilibrium point can now be determined from the Hamiltonian. To demonstrate this, the class of turning point of the Hamiltonian can be found from[5]

$$q(R) = \frac{\partial^2\mathcal{H}}{\partial R^2}\frac{\partial^2\mathcal{H}}{\partial\dot{R}^2} - \left[\frac{\partial^2\mathcal{H}}{\partial R\partial\dot{R}}\right]^2 = -\Omega^2 - \frac{2\mu}{R^3} \tag{6}$$

so that, substituting Eq. (5), $q(\bar{R}) = -3\Omega^2$, demonstrating that the equilibrium point is indeed hyperbolic. ($\mathcal{H}$ has a saddle point at $R = \bar{R}$.) The eigenvalues λ of the linear system in the neighborhood of the hyperbolic point E can also be determined from the characteristic polynomial P defined by[5]

$$P(\lambda) = \left\| \begin{bmatrix} \dfrac{\partial^2\mathcal{H}}{\partial R\partial\dot{R}} & \dfrac{\partial^2\mathcal{H}}{\partial\dot{R}^2} \\ -\dfrac{\partial^2\mathcal{H}}{\partial R^2} & -\dfrac{\partial^2\mathcal{H}}{\partial R\partial\dot{R}} \end{bmatrix}_{R=\bar{R}} - \lambda I_{2\times 2} \right\| = 0 \tag{7}$$

which, from Eq. (2), reduces to

$$\lambda = \pm\sqrt{-\frac{\partial^2\mathcal{H}}{\partial R^2}}\Bigg|_{R=\bar{R}} = \pm\sqrt{3}\Omega \tag{8}$$

This pair of real eigenvalues correspond to stable ($-\sqrt{3}\Omega$) and unstable ($+\sqrt{3}\Omega$) manifolds attached to E, shown in Fig. 2. Because E is a saddle point, it represents a potential barrier to particles attempting to transit it from either direction along the tether, represented by the transit and no-transit phase paths shown in Fig. 2. Level curves parameterized by C are shown in Fig. 3, where the equilibrium E corresponds to a hyperbolic fixed point. The stable and unstable manifolds are a linearization of the separatrix K, which discriminates globally between transit and no-transit paths.

Because the Hamiltonian of the problem forms an integral, the radial speed V of a particle moving freely along the tether can be obtained from Eq. (3) as

$$V(R)^2 = R^2\Omega^2 + 2\mu/R + 2C \tag{9}$$

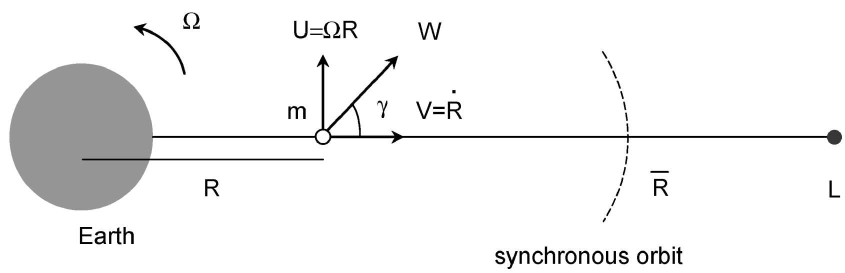

Fig. 1 Schematic orbital tower of length *L* corotating with angular velocity Ω.

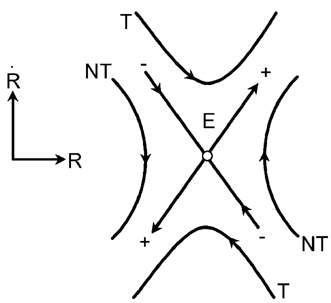

Fig. 2 Schematic hyperbolic fixed point E with stable (−) and unstable manifolds (+), transit (T) and no-transit (NT) phase paths.

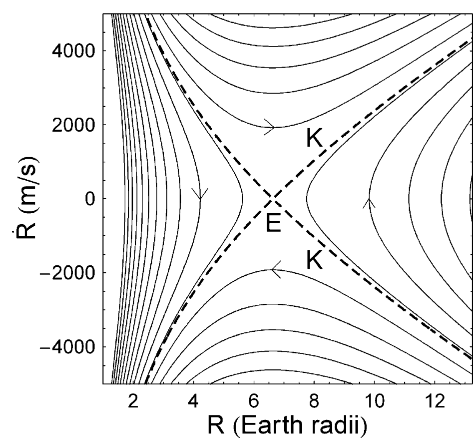

Fig. 3 Phase paths for particles moving along an Earth orbital tower with separatrix K.

while the transverse speed U is given by

$$U(R)^2 = R^2\Omega^2 \tag{10}$$

Therefore, the angle γ of the absolute velocity of the particle relative to the tether can be written as

$$\tan\gamma(R) = \frac{R\Omega}{\sqrt{R^2\Omega^2 + 2\mu/R + 2C}} \to 1, \qquad \gamma \to \frac{\pi}{4} \tag{11}$$

which yields the limit of $\pi/4$ for large R. Similarly, the absolute speed of the particle W can be obtained from Eqs. (9) and (10) as

$$W(R) = \sqrt{2R^2\Omega^2 + 2\mu/R + 2C} \to \sqrt{2}R\Omega \tag{12}$$

which yields a limit for large R. This result will be used later to assess the effect of friction on the launch performance of the tether.

For particles moving down the tether, after being captured at the end of the tether, the effective particle energy C must be such that the particle can cross the potential barrier at E and avoid being reflected back on a no-transit phase trajectory. Evaluating C at E, the following function can be defined:

$$f(\xi) = \sqrt{\Omega^2(\xi^2 - \bar{R}^2) + 2\mu(1/\xi - 1/\bar{R})} \tag{13}$$

for arbitrary ξ, such that to ensure transit of E, it is required that $-V(L) \succ f(L)$. Similarly, for particles ascending the tether under power, if $V(R) \succ f(R)$ then the particle will pass through E and escape with no further external energy required. Clearly, there are issues concerning the optimum strategy to ascend and descend the tower (minimum time, minimum energy); however, these are beyond the scope of this Note.

Particle Dynamics with Friction

Because most concepts for practical orbital towers require rapid transits of payloads along the tower, we assume that the coefficient of friction η (although uncertain) is small $\eta < 1$. The mechanism by which friction arises is through the transverse Coriolis force experienced by the particle as it transits the tower. For a mass m, the transverse force is $2m\Omega V$, and so the frictional force can be defined as $2\eta m\Omega V$, where η is the coefficient of friction. Therefore, Eq. (4) can be modified and rewritten as

$$\ddot{R} = R\Omega^2 - \mu/R^2 - 2\eta\Omega\dot{R} \tag{14}$$

which is no longer conservative. The equation of motion can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} R \\ \dot{R} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \Omega^2 & -2\eta\Omega \end{bmatrix}\begin{bmatrix} R \\ \dot{R} \end{bmatrix} + \begin{bmatrix} 0 \\ -\mu/R^2 \end{bmatrix} \tag{15}$$

so there is still a single equilibrium point at $R = \bar{R}$. The linear stability properties of the new system can be determined from the characteristic polynomial

$$P(\lambda) = \left\| \begin{bmatrix} 0 & 1 \\ 3\mu/R^3 & -2\eta\Omega \end{bmatrix}_{R=\bar{R}} - \lambda I_{2\times 2} \right\| = 0 \tag{16}$$

which reduces to

$$\lambda = \pm\sqrt{3 + \eta^2}\Omega - \eta\Omega \tag{17}$$

so that E remains a hyperbolic fixed point, as expected, although with a modified instability timescale.

In addition, numerical construction of the phase space shows that phase trajectories converge asymptotically to a single manifold S, as shown in Fig. 4. An approximation to this manifold can be developed to assess the effect of friction on the launch performance of the orbital tower. Eliminating time as the independent variable in Eq. (14), we obtain

$$\frac{\mathrm{d}V}{\mathrm{d}R} = \frac{1}{V}\left[R\Omega^2 - \frac{\mu}{R^2}\right] - 2\eta\Omega \tag{18}$$

which represents a nonlinear, first-order equation. As will be seen, Eq. (18) can be transformed to the class of Abel equations for solution. Notice that $\mathrm{d}V/\mathrm{d}R = 0$ when the radial speed $V = \bar{V}(R)$ is defined as

$$\bar{V}(R) = (1/2\eta\Omega)[R\Omega^2 - \mu/R^2] \tag{19}$$

which describes the curve M in phase space on which the particle has, instantaneously, null acceleration, as shown in Fig. 4. To provide

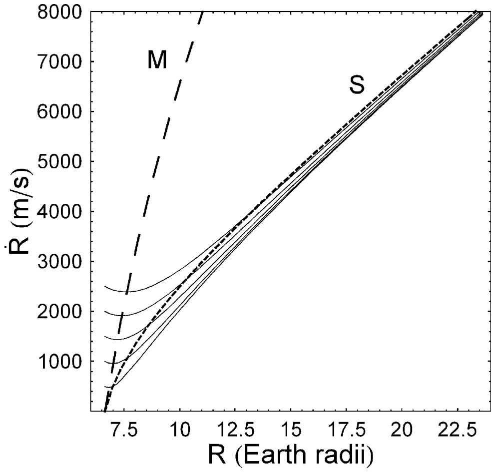

Fig. 4 Phase paths for particles moving along an Earth orbital tower with friction ($\eta = 0.25$), with null acceleration curve M and asymptotic manifold S.

a closed-form solution to Eq. (18), we transform the dependent variable to $Z(R) = V(R)^{-1}$ to obtain

$$\frac{\mathrm{d}Z}{\mathrm{d}R} - 2\eta\Omega Z^2 + \left[R\Omega^2 - \frac{\mu}{R^2}\right]Z^3 = 0 \qquad (20)$$

which forms one of the class of Abel equations. Because the phase paths converge asymptotically to a single manifold S, Eq. (20) will be used to construct an approximation to this manifold in the limit $R \succ\succ \bar{R}$. In this limit we assume that gravity can be neglected because the ratio of gravitational to centripetal acceleration scales as $(\bar{R}/R)^3$. With this assumption Eq. (20) can be solved to obtain the implicit relationship[6]

$$\frac{\eta}{\sqrt{1+\eta^2}}\tanh^{-1}\left(\frac{V/\Omega R + \eta}{\sqrt{1+\eta^2}}\right) + \frac{1}{2}\log(V^2 - \Omega^2 R^2 + 2\eta\Omega V R) = D \qquad (21)$$

for some constant of integration D. In the limit $(\bar{R}/R)^3 \succ\succ 1$, Eq. (21) can be approximated to obtain a simple form for the asymptote of the manifold S with the condition that $V(\bar{R}) = 0$ so that

$$V(R) \sim \sqrt{\Omega^2 R^2 (1+\eta^2) - \Omega^2 \bar{R}^2} - \eta\Omega R \qquad (22)$$

which, for small η, can be reduced to

$$V(R) \sim (1-\eta)\Omega R \qquad (23)$$

Equation (23) satisfies Eq. (18) to $\mathcal{O}(\eta)$, if the approximation that $(\bar{R}/R)^3 \succ\succ 1$ is again made. The approximation to the asymptote is shown in Fig. 4, from synchronous radius $\bar{R}$ to the end of the tower L (23.6 Earth radii) (Ref. 1). The reduction in speed of the particle caused by friction when it reaches the end of the tower at $R = L$ can therefore by written to $\mathcal{O}(\eta)$ as

$$\delta V(L) \sim \eta\Omega L \qquad (24)$$

because, from Eq. (9), the radial speed of the point mass along the tower scales as $V(R) \sim \Omega R$ for large R. The fractional reduction in launch performance of the tower therefore scales as $\delta V(L)/V(L) \sim \eta$. Because the manifold S is reached by all phase paths, the reduction in launch performance is essentially independent of particle speed at the transit of the fixed point E.

Conclusions

A simple analysis of the dynamics of a particle moving along an orbital tower has been presented. Using phase-space analysis, it is shown that a hyperbolic fixed point exists at synchronous radius and that the potential barrier formed by this fixed point leads to phase paths that do not transit the fixed point. This imposes a minimum capture speed for payloads descending the tower to avoid a no-transit phase path, which is reflected by the fixed point. The effect of friction, induced by the Coriolis acceleration, has been added, and an approximation to the asymptotic manifold in phase space has been constructed as a solution to an Abel-type equation, allowing an estimation of frictional losses for payloads being launched from the tower.

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